Speeding up MAP with Column Generation and Block Regularization

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Abstract

In this paper, we show how the connections between max-product message passing and linear programming relaxations for MAP allow for a more efficient exact algorithm than standard dynamic programming. Our proposed algorithm uses *column generation* to pass messages only on a small subset of the possible assignments to each variable, while guaranteeing to find the exact solution. This algorithm is more than two times faster than Viterbi decoding for part-of-speech tagging on WSJ data and equivalently fast as beam search with a beam of size two, while being exact.

The empirical performance of column generation depends on how quickly we can rule out entire sets of assignments to the edges of the chain, which is done by bounding the contribution of the pairwise factors to the score of the solution. This provides an opportunity at the intersection of inference and learning: at training time, we can regularize the model in a way that makes inference faster without changing its structure.

1. Introduction

Many basic tasks in natural language processing—part-of-speech tagging, named entity recognition, noun-phrase chunking, and others—are often approached with linear-chain conditional random fields (Lafferty et al., 2001). Maximum a posteriori (MAP) decoding in these graphical models is known to be $O(nk^2)$, where k is the number of labels for each token and n is the length of the sequence, which can be expensive especially when processing large amounts of data. Even though training is known to fail to converge when using approximate inference (Kulesza et al., 2007) and approximate search at test time can decrease accuracy, most actual implementations use beam search or other approximations instead of exact inference for performance reasons.

While there are known costs to these approximate approaches in general, most of the time exact solutions are actually recovered with approximate search, at a much smaller computational cost than standard exact inference. This is true because in most linear-chain models many tagging decisions can be safely made based on local information, and transition scores are only necessary to disambiguate between states that appear equally reasonable in the absence of contextual information. It is desirable to directly exploit this property by performing efficient local decoding and then expanding the transition factors only when necessary, while still returning exact results.

In this paper we show how an approach based on delayed column generation for the LP relaxation of the MAP decoding problem in linear chains leads to exactly this behavior, where most decisions are made locally and yet the inference process is provably exact. Moreover, as the performance of this search algorithm depends precisely on making local decisions, regularizing the model to minimize the magnitude of the transition scores leads to even faster inference, effectively learning to search faster.

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2. Delayed column generation in linear programs

Column generation is a method for exactly solving linear programs with a large number of variables. It works by restricting the problem to a subset of its variables[1] (implicitly setting the others to zero) and alternating between solving this *restricted linear program* and selecting which variables should be added to it, based on whether they could potentially improve the objective. If no variables can improve the objective, one has a primal-dual pair which is optimal for the original, unrestricted, linear program.

The keys to column generation, then, are the method by which variables are added to the linear program and the test that no additional variable could potentially improve the objective. This criterion, based on the notion of *reduced cost*, is the same criterion often used for pivoting in the simplex algorithm (Bertsimas & Tsitsiklis, 1997; Lubbecke & Desrosiers, 2004). The difference between the algorithms is that simplex relies on the primal variables being enumerated explicitly, while column generation leaves them implicitly defined and "generates" them only as needed. This does not guarantee that column generation will scale better than simplex, however, since in the worst case all variables will be added to the restricted primal problem.

Consider the general LP:

$$\textbf{max.} \quad c^T x \quad \textbf{s.t.} \quad Ax \leq b, \quad x \geq 0 \tag{1}$$

With corresponding Lagrangian:

$$L(x, \lambda) = c^T x + \lambda^t (b - Ax) = \sum_i \left(c_i - A_i^T \lambda\right) x_i + \lambda^t b \tag{2}$$

For a given assignment to the dual variables λ, a variable x_i is a candidate for being added to the restricted problem if its *reduced cost* $r_i = c_i - A_i^T \lambda$, the scalar multiplying it in the Lagrangian, is positive. Another way to justify this decision rule is by considering the constraints in the LP dual:

$$\textbf{min.} \quad b^T \lambda \quad \textbf{s.t.} \quad A^T \lambda \geq c \quad \lambda \geq 0 \tag{3}$$

Here, the reduced cost of a primal variable equals the degree to which its dual constraint is violated, and thus column generation in the primal is equivalent to cutting planes in the dual (Lubbecke & Desrosiers, 2004). Note that if there is no variable of positive reduced cost then the current dual variables from the restricted problem are feasible in the unrestricted problem, and thus we have a primal-dual optimal pair, and can terminate column generation.

[1]Here we refer to variables in the linear program, which are not the same as variables in the model.

3. Linear programming for MAP inference in chains

For any Markov random field whose graph is a tree the MAP inference problem can be formulated as a linear programming problem, where one maximizes the sum of the scores assigned by the model's factors to all variables and adjacent pairs of variables, subject to the assignment being contained in the marginal polytope, which for tree graphs is equivalent to marginals over nodes summing to one and marginals over edges being consistent with the node marginals (Wainwright & Jordan, 2008). Performing message-passing for max-product belief propagation on this graph can be shown to be equivalent to computing a set of optimal dual variables for this linear program. We exploit this connection to design an efficient algorithm for MAP inference based on column generation.

We focus on inference in models that are chain-structured, a special case of trees. A chain model over variables $V_1, \ldots, V_n$ can be expressed in terms of local factors $\theta_i(x_i)$, where x_i refers to a setting of variable i, and pairwise transition factors $\tau_i(x_i, x_{i+1})$. Following Wainwright & Jordan (2008), we write the MAP inference problem as the following LP:

$$\begin{array}{ll} \textbf{max.} & \sum_{i,x_i} \mu_i(x_i)\theta_i(x_i) \\ & + \sum_i \sum_{x_i, x_{i+1}} \mu_i(x_i, x_{i+1})\tau_i(x_i, x_{i+1}) \\ \textbf{s.t.} & \sum_{x_i} \mu_i(x_i) = 1 \\ & \sum_{x_i} \mu_i(x_i, x_{i+1}) = \mu_{i+1}(x_{i+1}) \\ & \sum_{x_{i+1}} \mu_i(x_i, x_{i+1}) = \mu_i(x_i) \end{array} \tag{4}$$

We refer to the first family of constraints as "normalization" constraints and the other two as "marginalization" constraints.

We can invoke the first marginalization constraint to combine the local and pairwise scores and replace the coefficient of $\mu_i(x_i, x_{i+1})$ in the objective with $(\tau_i(x_i, x_{i+1}) + \theta_{i+1}(x_{i+1}))$. Next, by relaxing the marginalization constraints (but keeping normalization as a hard constraint) and grouping all the terms according to the primal variables they multiply, we have the Lagrangian $L(\mu, \alpha, \beta)$, equal to

$$\sum_1 \mu_1(x_1)\nu_1(x_1) \tag{5}$$
$$+ \sum_{i, x_i, x_{i+1}} \mu_i(x_i, x_{i+1}) \left(\nu_i(x_i, x_{i+1}) - \nu_{i+1}(x_{i+1})\right)$$

This expression is a function of the max-marginals

$$\begin{aligned} \nu_i(x_i) &= \alpha_i(x_i) + \beta_i(x_i) + \theta_i(x_i) && (6) \\ \nu_i(x_i, x_{i+1}) &= \alpha_i(x_i) + \theta_i(x_i) + \tau_i(x_i, x_{i+1}) && (7) \\ &\quad +\theta_{i+1}(x_{i+1}) + \beta_{i+1}(x_{i+1}). \end{aligned}$$

Here, $\alpha_i(x_i)$ and $\beta_i(x_i)$ are dual variables corresponding to the marginalization constraints when variable i is set to x_i. Wainwright & Jordan (2008) show that the fixed point of the max-product message passing rules provides an assignment to the dual variables α and β that, together with decoding based on the max-marginals form a primal-dual optimal pair for this LP. The update equations for these messages are as follows:

$$\alpha_{i+1}(x_{i+1}) = \max_{x_i} \alpha_i(x_i) + \theta_i(x_i) + \tau(x_i, x_{i+1}) \quad (8)$$

$$\beta_{i-1}(x_{i-1}) = \max_{x_i} \tau(x_{i-1}, x_i) + \theta_i(x_i) + \beta_i(x_i). \quad (9)$$

Note that in the Lagrangian of equation (5) we made an arbitrary decision to single out the first node in the chain. An equivalent decision can be made by choosing the last node, which will be important in the next section.

4. A column-generation algorithm for fast chain decoding

In many applications of MAP inference in chains, such as in text sequence tagging, the graphical model variables have large domain sizes. Let D be the number of values that x_i can take on and n be the length of the chain. The LP in Equation (4) would have $O(nD^2)$ variables. Most of the time, however, we can be confident that many of these variables will not be used in the final solution, due to the strength of local factors $\theta_i(x_i)$ relative to the pairwise factors $\tau_i(x_i, x_{i+1})$. We leverage this property by lazily adding variables to the LP using column generation.

4.1. Deriving the algorithm

We seek to solve the LP for MAP inference in chains with a column generation strategy. Such a strategy requires efficient components for choosing the initial set of variables in the restricted LP, solving the restricted LP, and finding variables of positive reduced cost.

To initialize the LP, we first define for each node in the graphical model a restricted domain consisting of only $x_i = \text{argmax}\, \theta_i(x_i)$. Note that any initialization strategy is equally valid, and one could, for example, also add the high-scoring transitions, or add the k best x_i instead of the single best. Next, we include in the initial restricted LP all the $\mu_i(x_i)$ and $\mu_i(x_i, x_{i+1})$ indicator variables corresponding to these size-one domains.

To solve the restricted LP, we use max-product message passing, but adapt the recursive definition of $\alpha_i(x_i)$ and $\beta_i(x_i)$ in equations (8) and (9) to only search over the restricted domain of a variable's neighbors in the chain. This can be derived by looking at the Lagrangian of the LP with all $\mu_i(x_i)$ variables but only a subset of the $\mu_i(x_i, x_{i+1})$ variables, but we omit this derivation due to space constraints.

For column generation, we need to compute the reduced costs of each pairwise marginal variable $\mu_i(x_i, x_{i+1})$ in terms of θ, τ, α, and β. By using the Lagrangian from equation (5) we get the following reduced cost for the pairwise marginals

$$\begin{aligned} R'_i(x_i, x_{i+1}) &= \nu_i(x_i, x_{i+1}) - \nu_{i+1}(x_{i+1}) \\ &= \tau(x_i, x_{i+1}) + \theta_i(x_i) && (10) \\ &\quad +\alpha_i(x_i) - \alpha_{i+1}(x_{i+1}). \end{aligned}$$

When we solved the restricted LP, we didn't allow $x_i \notin D_i$ to take on nonzero values. However, we still needed to assign values to the dual variables indexed by x_i. We set $\alpha_i(x_i)$ as defined in equation 8, even for $x_i \notin D_i$. Doing so does not interfere with the dual optimality of the messages passed in the restricted problem.

Equation (10) has a simple interpretation. $\alpha_{i+1}(x_{i+1})$ is defined as a max over the current restricted domain D_i. For $x_i \notin D_i$, the first three terms of R'_i represent the value of $\alpha_{i+1}(x_{i+1})$, had this maximization used x_i. Therefore, $R'_i(x_i, x_{i+1})$ represents the gain in forward message $\alpha_{i+1}(x_{i+1})$ attainable if x_i is added to restricted domain D_i.

Because this reduced cost only involves α variables, it only considers one direction in the chain when judging the desirability of a pair of variables. As described in section 3, our Lagrangian has an inherent asymmetry where V_1 was chosen as the root of the chain. If we had chosen V_n as the root, our reduced cost would only use β information. Both of these Lagrangians come from the same original LP, and have the same optimum. Therefore, we can average them in order to obtain a Lagrangian for which the reduced cost contains global information from both directions. Doing so leads us to the expression for the reduced cost we'll use in the remainder of this paper,

$$\begin{aligned} 2R_i(x_i, x_{i+1}) &= 2\tau(x_i, x_{i+1}) + \theta_i(x_i) + \theta_{i+1}(x_{i+1}) \\ &\quad + (\alpha_i(x_i) - \alpha_{i+1}(x_{i+1})) \\ &\quad + (\beta_{i+1}(x_{i+1}) - \beta_i(x_i)). && (11) \end{aligned}$$

Algorithm 1 Column Generation Chain MAP

for $i = 1 \rightarrow n$ **do** $D_i = \{\text{argmax}\, \theta_i(x_i)\}$
end for
while domains haven't converged **do**
  $(\alpha, \beta) \leftarrow GetMessages(D, \theta)$
  for $i = 1 \rightarrow n$ **do**
    $(x_i, x_{i+1}, rc) \leftarrow \text{argmax}\, R_i(x_i, x_{i+1})$
    if $rc > 0$ **then**
      $D_i \leftarrow D_i \cup x_i$
      $D_{i+1} \leftarrow D_{i+1} \cup x_{i+1}$
    end if
  end for
end while

Algorithm 2 Efficient search for a setting with positive reduced cost

$U_\tau(\cdot, x_{i+1}) \leftarrow \max_{x_i} \tau(x_i, x_{i+1})$
$U_\tau(x_i, \cdot) \leftarrow \max_{x_{i+1}} \tau(x_i, x_{i+1})$
$U_i \leftarrow \max_{x_i} N_i(x_i)$
$C'_i \leftarrow \{x_{i+1} | N'_i(x_{i+1}) + U_i + 2U_\tau(\cdot, x_{i+1}) > 0\}$
$U'_i \leftarrow max_{x_{i+i} \in C'_i} N'_i(x_{i+1})$
$C_i \leftarrow \{x_i | N_i(x_i) + U'_i + 2U_\tau(x_i, \cdot) > 0\}$
$(x^*_i, x^*_{i+1}) \leftarrow \text{argmax}_{x_i \in C_i, x_{i+1} \in C'_i} R(x_i, x_{i+1})$

At every iteration of our column generation procedure, we find for each position i in the chain the setting of x_i and x_{i+1} that maximizes the reduced cost $R_i(x_i, x_{i+1})$. If that reduced cost is positive, that setting of the variables is added to our domain. If no setting with a positive reduced cost was found we can prove that we have an optimal solution. Algorithm 1 shows the pseudocode for this approach.

4.2. Finding a state with positive reduced cost

Naively searching for pairwise variables of positive reduced cost has the same $O(nk^2)$ complexity as standard Viterbi inference. Therefore, pruning is necessary when performing this search.

Note that the terms in the reduced cost equation (11) can be divided into three groups: those that depend only on x_i, $N_i(x_i) = \theta_i(x_i) + \alpha_i(x_i) - \beta_i(x_i)$, those that depend only on x_{i+1} (henceforth $N'_i(x_{i+1})$) and the transition scores $\tau_i(x_i, x_{i+1})$. Henceforth we assume the transition scores are the same across locations in the chain, though it is trivial to generalize to the case where they vary, and it is possible to obtain tighter bounds while doing that.

Given the decomposition above of the reduced cost we can develop an efficient searching strategy based on independent bounds as in Algorithm 2.

This strategy effectively reduces the common-case complexity of the search for settings with positive reduced cost from $O(k^2)$ to $O(k)$. Note that this strategy can be inefficient if the bounds $U_\tau(x_i, \cdot)$ and $U_\tau(\cdot, x_{i+1})$ on the rows and columns of the transition matrix are loose, which suggests that, during learning, we should try to keep the transition scores as small as possible.

This is the source of our connection between inference and learning: by learning to keep the transition scores small we can effectively learn to search faster. Therefore, we can obtain an accuracy-speed tradeoff of our decoding algorithm using *block regularization*, where we apply different regularization strategies to the transition scores and emission scores when training.

5. Related Work

Column generation is a cutting plane algorithm applied to the dual problem. Cutting planes have been successfully applied to problems such as training structured SVMs and improving approximate MAP inference in loopy graphical models (Tsochantaridis et al., 2006; Sontag & Jaakkola, 2007). While column generation has enabled solutions to key operations research problems, such as those by Gilmore & Gomory (1961) and Desrochers & Soumis (1989), it is not widely known in machine learning.

Most related work in methods for decoding chains has focussed on methods for improving the accuracy of approximate methods, such as beam search. For example, Pal et al. (2006) selected an adaptive beam width at every variable by ensuring that the marginal distribution captured by the beam was sufficiently close to the true marginal distribution of the chain variable.

Some prior work has also improved exact decoding of linear chains. Kaji et al. (2010) presents an efficient decoding strategy by dividing the set of values for each node in the chain into two groups, and computing the messages explicitly for values in one group while upper-bounding the messages for values in the other group. Esposito & Radicioni (2007) offered the CarpeDiem algorithm, which lazily grows the domains of output variables using bounds on various model scores.

6. Experiments

We perform two sets of experiments. The first explores the tradeoff between computation time and exactness, by comparing our column-generation approach to beam search and exact Viterbi inference. The second explores the relationship between regularizing the transition weights and inference efficiency. We trade off accuracy against time in a characteristically differ-

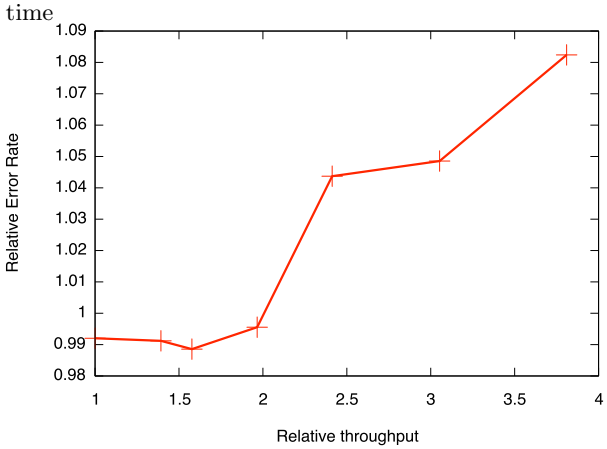
Table 1. Timing experiments for joint POS and NER

Algorithm	sentences/s	% exact
Viterbi	57	100
Column generation	779	100
Beam-1	3717	66.6
Beam-5	995	99.5
Beam-7	772.8	99.7
Beam-10	575.1	99.9

Table 2. Timing experiments for POS.

Algorithm	sentences/s	% exact
Viterbi	3144	100
Column generation	8227	100
Beam-1	12117.6	57.7
Beam-2	7519	92.6
Beam-3	6802	98.4
Beam-4	5731	99.5

Figure 1. The tradeoff between accuracy and inference time



ent fashion than beam search, by learning a model in which it is easier to search.

6.1. Exploring the exactness versus time tradeoff

We compare the speed of our algorithm on a standard part-of-speech tagging task on Wall Street Journal data (Marcus et al., 1993) (WSJ) and on a joint part-of-speech tagging and named-entity recognition task on CoNLL 2003 data.

Table 1 shows the results for joint POS and NER. The model is a factorial conditional random field with one linear chain for NER and another for POS tagging and factors connecting both these chains to each other and to observed data. Inference was performed on the cluster graph for this model, in which each variable can take one out of 360 possible values (45 POS tags $\times$ 8 NER labels). Column generation is about 13 times faster than Viterbi, and as fast as Beam-7. Whereas our algorithm is exact, Beam-7 returned the MAP setting for 99.7% of the sentences.

Table 2 shows the results for the POS tagging experiment. The model was trained with 50 iterations of perceptron on the training set of the Penn Treebank. Note that the column generation algorithm processed more sentences per second than beam search with a beam of size two, which only found the exact MAP setting 92.6% of the time.

6.2. Exploring the regularization strength versus time tradeoff

Next, we perform a simple training experiment to illustrate the effect of regularizing the transition weights on inference time. We use a standard part-of-speech tagging conditional random field on for WSJ data trained with the structured SVM algorithm with ℓ_2 regularization (Tsochantaridis et al., 2006). We place a regularization coefficient of 0.1 on the emission weights and vary the coefficient on the transition weights from 0.1 to 10. Results are in Figure 1. A 4x gain in speed can be obtained at the expense of an 8% relative decrease in accuracy. With balanced block regularization, the model obtains a tagging accuracy of 96.4%.

7. Conclusions and future work

We present an efficient algorithm for MAP decoding in linear chains that uses the LP relaxation of the inference problem to adaptively prune the state-space necessary for exact inference. We also show how adapting the learning procedure can lead to further speed benefits of about 4x for a small relative decrease in accuracy. It is easy to generalize this approach to trees, and in that case instead of averaging two Lagrangians it is more natural to average one Lagragian per leaf. This algorithm can also be applied, for example, as a subroutine in tree block coordinate descent (Sontag & Jaakkola, 2009), leading to speed improvements without any change in the algorithm's guarantees.

A limitation of this algorithm is that it currently supports only local and transition factors. To generalize it to factors over trigrams, four-grams, etc., we only obtain a reduction in complexity from $O(k^d)$ to $O(k^{d-1})$, which, while significant, is still not practical for most purposes. However, perhaps by exploiting solutions to the bigram chain decoding model as initialization to higher-order models it might be possible to break this barrier, and effectively learn exact-inference analogues to structured prediction cascades (Weiss & Taskar, 2010), in which much of the hypothesis space is pruned by lower-order models.

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